# Solutions to tutorial exercises for stochastic processes

T1. We will prove the statement by induction on $n$. The induction base is exactly the Feller property. Now suppose

$$x \mapsto \mathbb{E}^x \prod_{k=1}^{n} f_k(X_{t_k})$$

is an element of $C_0(S)$. For $n+1$ we can use the tower property and the Markov property to obtain

$$\begin{aligned} E^x\left[\prod_{k=1}^{n+1} f_k(X_{t_k})\right] &= E^x\left[\prod_{k=1}^{n} f_k(X_{t_k})\mathbb{E}^x\left[f_{n+1}(X_{t_{n+1}}) \;\middle|\; \mathfrak{F}_{t_n}\right]\right] \\ &= E^x\left[\prod_{k=1}^{n} f_k(X_{t_k})\mathbb{E}^{X_{t_n}}\left[f_{n+1}(X_{t_{n+1}-t_n})\right]\right]. \end{aligned}$$

Let

$$g(x) = \mathbb{E}^x\left[f_{n+1}(X_{t_{n+1}-t_n})\right],$$

then $g(x) \in C_0(S)$ by the Feller property. Therefore $(f_n g)(x) \in C_0(S)$. We conclude that

$$E^x\left[\prod_{k=1}^{n+1} f_k(X_{t_k})\right] = E^x\left[\prod_{k=1}^{n-1} f_k(X_{t_k})(f_n g)(X_{t_n})\right]$$

is an element of $C_0(S)$ by the induction hypothesis.

T2. (a) We only prove property (S2) of the probability semigroup, the other properties were proven in the lecture. We have

$$\begin{aligned} \|T_t f - f\| = \sup_{x\in S}\left|(T_t f)(x) - f(x)\right| &= \sup_{x\in S}\left|\mathbb{E}^0\left[f(x+B_t)\right] - f(x)\right| \\ &\le \mathbb{E}^0\left[\sup_{x\in S}\left|f(x+B_t) - f(x)\right|\right]. \end{aligned}$$

Since $f \in C_0(S)$, it vanishes at infinity, and it is therefore uniformly continuous. This implies that

$$\sup_{x\in S}\left|f(x+B_t) - f(x)\right| \to 0 \quad \text{as} \quad t \downarrow 0 \quad \text{a.s.},$$

since $B_t \to 0$ almost surely. It follows by the dominated convergence theorem that $\|T_t f - f\| \to 0$ as $t \downarrow 0$.

(b) If $f \in C_b(S)$ it is not necessarily uniformly continuous, so that the above argument does not hold. For example consider $f(x) = \max\{\cos(x^2), 0\}$. Then it can be shown that for every $t > 0$ there exists an $x \in S$ such that $\mathbb{E}^0[f(x + B_t)] - f(x)]$ is bounded away from zero independent of $t$, so that $T_t f$ does not converge to $f$.

T3. (G1): Firstly, $\mathcal{D}(\mathcal{L})$ is a vector space. To use the Stone-Weierstrass theorem we further need to show that $\mathcal{D}(\mathcal{L})$ separates points and vanishes nowhere. Consider the functions $f_a(x) = \exp(-(x-a)^2) \in \mathcal{D}(\mathcal{L})$. Then for all pairs $x \neq y$ in $S$ we have $f_x(x) = 1$ and $f_x(y) < 1$, so that $\mathcal{D}(\mathcal{L})$ separates points. Furthermore since $f_x(x) = 1$, the space vanishes nowhere. The theorem now states that $\mathcal{D}(\mathcal{L})$ is dense in $C_0(S)$.

(G2): Let $\lambda > 0$ and $g = f - \lambda f'$. Since $f \in C_0(\mathbb{R})$ we have $\inf_x f(x) \leq 0$. Similarly $\inf_x g(x) \leq 0$. If $\inf_x f(x) = 0$ we immediately have $\inf_x g(x) \leq \inf_x f(x)$. Now suppose $\inf_x f(x) < 0$, then since $f$ is continuous there exists $x_0 \in S$ with $f(x_0) = \inf_x f(x)$ and $f'(x_0) = 0$. We now get

$$\inf_x f(x) = f(x_0) = f(x_0) - \lambda f'(x_0) \geq \inf_x g(x).$$

(G3): Let $g \in C_0(s)$. We need to show that there exists an $f \in \mathcal{D}(\mathcal{L})$ with $f - \lambda f' = g$. This differential equation is solved by

$$f(x) = Ce^{\frac{1}{\lambda}x} - \int_0^x \frac{1}{\lambda} g(y) e^{\frac{1}{\lambda}(x-y)} \mathrm{d}y.$$

To make the computations easier we take

$$C = \int_0^\infty \frac{1}{\lambda} g(y) e^{-\frac{1}{\lambda}y},$$

so that

$$f(x) = \int_x^\infty \frac{1}{\lambda} g(y) e^{\frac{1}{\lambda}(x-y)} \mathrm{d}y.$$

We need to show that $f \in C_0(\mathbb{R})$. Continuity follows immediately, so it remains to show that $f$ vanishes at infinity. We have

$$|f(x)| \leq \frac{1}{\lambda} \sup_{y \in [x,\infty)} |g(y)| \lambda \to 0 \quad \text{as} \quad x \to \infty.$$

For the other limit we can write

$$f(x) = -\int_{-\infty}^0 \frac{1}{\lambda} g(x-y) e^{\frac{1}{\lambda}y} \mathrm{d}y.$$

The integrand is bounded by $\frac{1}{\lambda}\|g\|$, so that by dominated convergence

$$\lim_{x \to -\infty} |f(x)| = \left| \int_{-\infty}^0 \lim_{x \to -\infty} \frac{1}{\lambda} g(x-y) e^{\frac{1}{\lambda}y} \mathrm{d}y \right| = 0.$$

(G4): Let $\lambda > 0$. Consider $f_n(x) = \exp\left(\frac{-x^2}{n}\right)$, and

$$g_n(x) = f_n(x) - \lambda f_n'(x) = \exp\left(\frac{-x^2}{n}\right) + \frac{2\lambda x}{n}\exp\left(\frac{-x^2}{n}\right).$$

Then $\sup_n \|g_n\| < \infty$, $f_n \to 1$ and $g_n \to 1$ pointwise as $n \to \infty$.

This belongs to the process that moves deterministically to the right at unit speed: $X_t = X_0 + t$. The semigroup of this process is given by

$$(T_t f)(x) = \mathbb{E}^x[f(X_t)] = f(x+t).$$

This process indeed has generator $f'$:

$$\lim_{t \downarrow 0} \frac{(T_t f)(x) - f(x)}{t} = \lim_{t \downarrow 0} \frac{f(x+t) - f(x)}{t} = f'(x).$$